Elementary Set Theory
Part 2:
Set Operation, Cartesian Product, Inclusion-Exclusion Principle, and Set Partition

Mathematical Logic – First Term 2023-2024

MZI

School of Computing
Telkom University

FIF Tel-U

December 2023 – January 2024

Acknowledgements

This slide is compiled using the materials in the following sources:

1. *Discrete Mathematics and Its Applications* (Chapter 2), 8th Edition, 2019, by K. H. Rosen (primary reference).
2. *Discrete Mathematics with Applications* (Chapter 6), 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Discrete Mathematics 1 (2012) slides at Fasilkom UI by B. H. Widjaja.
5. Discrete Mathematics 1 (2010) slides at Fasilkom UI by A. A. Krisnadhi.

Some figures are excerpted from those sources. This slide is intended for internal academic purpose in SoC Telkom University. No slides are ever free from error nor incapable of being improved. Please convey your comments and corrections (if any) to `<pleasedontspam>@telkomuniversity.ac.id`.

Contents

Contents

Some Elementary Set Operations

Definition

Let A and B be two sets.

1. Union of A and B, denoted by $A \cup B$, is defined as $A \cup B := \{x \mid x \in A \textbf{ or } x \in B\}$ or $A \cup B := \{x \mid (x \in A) \vee (x \in B)\}$;
2. Intersection of A and B, denoted by $A \cap B$, is defined as $A \cap B := \{x \mid x \in A \textbf{ and } x \in B\}$ or $A \cap B := \{x \mid (x \in A) \wedge (x \in B)\}$; if $A \cap B = \emptyset$, then A and B are *disjoint* and we can write $A//B$.
3. Difference of A and B, denoted by $A \setminus B$, $A \smallsetminus B$, or $A - B$, is defined as $A \smallsetminus B := \{x \mid x \in A \textbf{ and } x \notin B\}$ or $A \smallsetminus B := \{x \mid (x \in A) \wedge (x \notin B)\}$;
4. *Xor or symmetric difference* of A and B, denoted by $A \oplus B$, is defined as $A \oplus B := \{x \mid (x \in A) \oplus (x \in B)\}$. Thus, we also have $A \oplus B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Definition

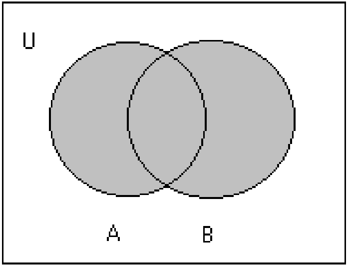
If A is considered over the universal set U, then the complement of A (with respect to U), denoted by $A^{'}$, A^C, $\bar{A}$, or $U \setminus A$, is defined as
$A^C := \{x \in U \mid x \notin A\}$.

$A \cup B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \cup B = \{x \in U : (x \in A) \vee (x \in B)\}$. The Venn diagram for $A \cup B$ is illustrated as follows.

$A \cup B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \cup B = \{x \in U : (x \in A) \vee (x \in B)\}$. The Venn diagram for $A \cup B$ is illustrated as follows.



Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \cup B =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
2. $A \cup \emptyset =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
2. $A \cup \emptyset = \{1, 2, 3, 4\}$.
3. $A \cup U =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

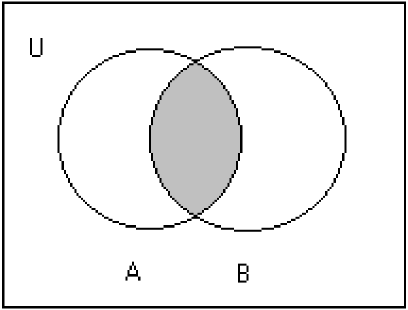
1. $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
2. $A \cup \emptyset = \{1, 2, 3, 4\}$.
3. $A \cup U = U$.

$A \cap B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \cap B = \{x \in U : (x \in A) \wedge (x \in B)\}$. The Venn diagram for $A \cap B$ is illustrated as follows.

$A \cap B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \cap B = \{x \in U : (x \in A) \wedge (x \in B)\}$. The Venn diagram for $A \cap B$ is illustrated as follows.



Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$, then:

1. $A \cap B =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$, then:

1. $A \cap B = \{3, 4\}$.
2. $B \cap C =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$, then:

1. $A \cap B = \{3, 4\}$.
2. $B \cap C = \{5, 6\}$.
3. $A \cap C =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$, then:

1. $A \cap B = \{3, 4\}$.
2. $B \cap C = \{5, 6\}$.
3. $A \cap C = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in C)$. Thus, we can write $A//C$ (A and C are disjoint).
4. $A \cap B \cap C =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$, then:

1. $A \cap B = \{3, 4\}$.
2. $B \cap C = \{5, 6\}$.
3. $A \cap C = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in C)$. Thus, we can write $A//C$ (A and C are disjoint).
4. $A \cap B \cap C = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in B) \wedge (x \in C)$. Observe that :
   $(A \cap B) \cap C = \{3, 4\} \cap \{5, 6, 7, 8\} = \emptyset$ and
   $A \cap (B \cap C) = \{1, 2, 3, 4\} \cap \{5, 6\} = \emptyset$. However, we don't write $A//B//C$ because A and B are not disjoint, so are B and C.
5. $A \cap \emptyset =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$, then:

1. $A \cap B = \{3, 4\}$.
2. $B \cap C = \{5, 6\}$.
3. $A \cap C = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in C)$. Thus, we can write $A//C$ (A and C are disjoint).
4. $A \cap B \cap C = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in B) \wedge (x \in C)$. Observe that : $(A \cap B) \cap C = \{3, 4\} \cap \{5, 6, 7, 8\} = \emptyset$ and $A \cap (B \cap C) = \{1, 2, 3, 4\} \cap \{5, 6\} = \emptyset$. However, we don't write $A//B//C$ because A and B are not disjoint, so are B and C.
5. $A \cap \emptyset = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in \emptyset)$. The truth value of $x \in \emptyset$ is always F.
6. $A \cap U =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$, then:

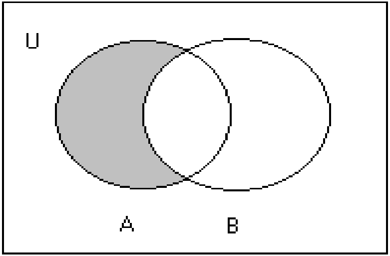
1. $A \cap B = \{3, 4\}$.
2. $B \cap C = \{5, 6\}$.
3. $A \cap C = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in C)$. Thus, we can write $A//C$ (A and C are disjoint).
4. $A \cap B \cap C = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in B) \wedge (x \in C)$. Observe that : $(A \cap B) \cap C = \{3, 4\} \cap \{5, 6, 7, 8\} = \emptyset$ and $A \cap (B \cap C) = \{1, 2, 3, 4\} \cap \{5, 6\} = \emptyset$. However, we don't write $A//B//C$ because A and B are not disjoint, so are B and C.
5. $A \cap \emptyset = \emptyset$, because there is no $x \in U$ such that $(x \in A) \wedge (x \in \emptyset)$. The truth value of $x \in \emptyset$ is always F.
6. $A \cap U = A$, because if $x \in A \cap U$ then x satisfies $(x \in A) \wedge (x \in U)$, therefore $x \in A$, and so $A \cap U = A$.

$A \setminus B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \setminus B = \{x \in U : (x \in A) \wedge (x \notin B)\} = \{x \in U : (x \in A) \wedge \neg(x \in B)\}$. The Venn diagram for $A \setminus B$ is illustrated as follows.

$A \setminus B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \setminus B = \{x \in U : (x \in A) \wedge (x \notin B)\} = \{x \in U : (x \in A) \wedge \neg (x \in B)\}$. The Venn diagram for $A \setminus B$ is illustrated as follows.



Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \setminus B =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \setminus B = \{1, 2, 3, 4\} \setminus \{3, 4, 5, 6\} = \{1, 2\}$.
2. $B \setminus A =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, then:

1. $A \setminus B = \{1,2,3,4\} \setminus \{3,4,5,6\} = \{1,2\}$.
2. $B \setminus A = \{3,4,5,6\} \setminus \{1,2,3,4\} = \{5,6\}$.
3. $U \setminus A =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, then:

1. $A \setminus B = \{1,2,3,4\} \setminus \{3,4,5,6\} = \{1,2\}$.
2. $B \setminus A = \{3,4,5,6\} \setminus \{1,2,3,4\} = \{5,6\}$.
3. $U \setminus A = \{x \in \mathbb{N} : x \leq 10\} \setminus \{1,2,3,4\} = \{x \in U : (x \in U) \wedge (x \notin A)\} = \{5,6,7,8,9,10\} = A^C$.
4. $A \setminus U =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \setminus B = \{1, 2, 3, 4\} \setminus \{3, 4, 5, 6\} = \{1, 2\}$.
2. $B \setminus A = \{3, 4, 5, 6\} \setminus \{1, 2, 3, 4\} = \{5, 6\}$.
3. $U \setminus A = \{x \in \mathbb{N} : x \leq 10\} \setminus \{1, 2, 3, 4\} = \{x \in U : (x \in U) \wedge (x \notin A)\} = \{5, 6, 7, 8, 9, 10\} = A^C$.
4. $A \setminus U = \{1, 2, 3, 4\} \setminus \{x \in \mathbb{N} : x \leq 10\} = \{x \in U : (x \in A) \wedge (x \notin U)\} = \emptyset$.
5. $A \setminus \emptyset =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, then:

1. $A \setminus B = \{1,2,3,4\} \setminus \{3,4,5,6\} = \{1,2\}$.
2. $B \setminus A = \{3,4,5,6\} \setminus \{1,2,3,4\} = \{5,6\}$.
3. $U \setminus A = \{x \in \mathbb{N} : x \leq 10\} \setminus \{1,2,3,4\} = \{x \in U : (x \in U) \wedge (x \notin A)\} = \{5,6,7,8,9,10\} = A^C$.
4. $A \setminus U = \{1,2,3,4\} \setminus \{x \in \mathbb{N} : x \leq 10\} = \{x \in U : (x \in A) \wedge (x \notin U)\} = \emptyset$.
5. $A \setminus \emptyset = \{1,2,3,4\} \setminus \emptyset = \{x \in U : (x \in A) \wedge (x \notin \emptyset)\} = \{1,2,3,4\}$, because $x \notin \emptyset$ is always T.
6. $\emptyset \setminus A =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

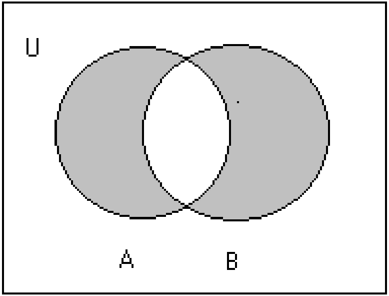
1. $A \setminus B = \{1, 2, 3, 4\} \setminus \{3, 4, 5, 6\} = \{1, 2\}$.
2. $B \setminus A = \{3, 4, 5, 6\} \setminus \{1, 2, 3, 4\} = \{5, 6\}$.
3. $U \setminus A = \{x \in \mathbb{N} : x \leq 10\} \setminus \{1, 2, 3, 4\} = \{x \in U : (x \in U) \wedge (x \notin A)\} = \{5, 6, 7, 8, 9, 10\} = A^C$.
4. $A \setminus U = \{1, 2, 3, 4\} \setminus \{x \in \mathbb{N} : x \leq 10\} = \{x \in U : (x \in A) \wedge (x \notin U)\} = \emptyset$.
5. $A \setminus \emptyset = \{1, 2, 3, 4\} \setminus \emptyset = \{x \in U : (x \in A) \wedge (x \notin \emptyset)\} = \{1, 2, 3, 4\}$, because $x \notin \emptyset$ is always T.
6. $\emptyset \setminus A = \emptyset \setminus \{1, 2, 3, 4\} = \{x \in U : (x \in \emptyset) \wedge (x \notin A)\} = \emptyset$, because $x \in \emptyset$ is always F.

$A \oplus B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) = \{x \in U : (x \in A) \oplus (x \in B)\}$. The Venn diagram for $A \oplus B$ is illustrated as follows.

$A \oplus B$

Suppose A and B are two sets considered over the universal set U. Then, we have $A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) = \{x \in U : (x \in A) \oplus (x \in B)\}$. The Venn diagram for $A \oplus B$ is illustrated as follows.



Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \oplus B =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \oplus B = \{1, 2, 5, 6\}$, $B \oplus A =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \oplus B = \{1, 2, 5, 6\}$, $B \oplus A = \{1, 2, 5, 6\}$.
2. $A \oplus U =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $A \oplus B = \{1, 2, 5, 6\}$, $B \oplus A = \{1, 2, 5, 6\}$.
2. $A \oplus U = \{x \in U : (x \in A) \oplus (x \in U)\} = (A \cup U) \setminus (A \cap U) = U \setminus A = \{5, 6, 7, 8, 9, 10\} = A^C$.
3. $A \oplus \emptyset =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

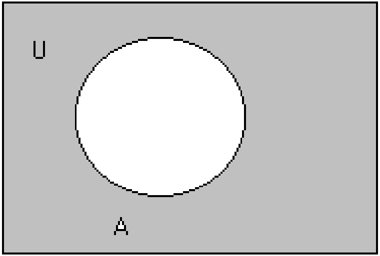
1. $A \oplus B = \{1, 2, 5, 6\}$, $B \oplus A = \{1, 2, 5, 6\}$.
2. $A \oplus U = \{x \in U : (x \in A) \oplus (x \in U)\} = (A \cup U) \setminus (A \cap U) = U \setminus A = \{5, 6, 7, 8, 9, 10\} = A^C$.
3. $A \oplus \emptyset = \{x \in U : (x \in A) \oplus (x \in \emptyset)\} = (A \cup \emptyset) \setminus (A \cap \emptyset) = A \setminus \emptyset = \{1, 2, 3, 4\} = A$.

A', A^C, or $\bar{A}$

Suppose A is a set considered over the universal set U. We have $A' = \bar{A} = A^C = \{x \in U : x \notin A\}$. The Venn diagram for A^C is illustrated as follows

A', A^C, or $\bar{A}$

Suppose A is a set considered over the universal set U. We have $A' = \bar{A} = A^C = \{x \in U : x \notin A\}$. The Venn diagram for A^C is illustrated as follows



Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $\bar{A} =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $\bar{A} = \{x \in U : x \notin A\} = \{5, 6, 7, 8, 9, 10\}$.
2. $\bar{B} =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $\bar{A} = \{x \in U : x \notin A\} = \{5, 6, 7, 8, 9, 10\}$.
2. $\bar{B} = \{x \in U : x \notin B\} = \{1, 2, 7, 8, 9, 10\}$.
3. $\bar{U} =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $\bar{A} = \{x \in U : x \notin A\} = \{5, 6, 7, 8, 9, 10\}$.
2. $\bar{B} = \{x \in U : x \notin B\} = \{1, 2, 7, 8, 9, 10\}$.
3. $\bar{U} = \{x \in U : x \notin U\} = \emptyset$.
4. $\bar{\emptyset} =$

Example

Given the universal set $U = \{x \in \mathbb{N} : x \leq 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, then:

1. $\bar{A} = \{x \in U : x \notin A\} = \{5, 6, 7, 8, 9, 10\}$.
2. $\bar{B} = \{x \in U : x \notin B\} = \{1, 2, 7, 8, 9, 10\}$.
3. $\bar{U} = \{x \in U : x \notin U\} = \emptyset$.
4. $\bar{\emptyset} = \{x \in U : x \notin \emptyset\} = U$.

Set Identities

Suppose A, B, and C are sets considered over the universal set U.

$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$\left(A^C\right)^C = A$	Complementation laws

$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$(A \cup B)^C = A^C \cap B^C$ $(A \cap B)^C = A^C \cup B^C$	De Morgan laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup A^C = U$ $A \cap A^C = \emptyset$	Complement laws

From propositional logic course, we can deduce following analogy.

In set	In logic
$\cap$	$\wedge$
$\cup$	$\vee$
$\oplus$	$\oplus$
$(\cdots)^C$ or $\overline{(\cdots)}$	$\neg$
$\subseteq$	$\rightarrow$
$=$	$\leftrightarrow$
U (universal set)	T
$\emptyset$ (empty set)	F

We also have the following analogy: $\subseteq$ with $\leq$, $\subset$ with $<$, $\supseteq$ with $\geq$, and $\supset$ with $>$, where $\leq$, $<$, $\geq$, and $>$ are relational operators for numbers.

Furthermore, since $\oplus$ in propositional logic is commutative and associative, we obtain following theorem.

Theorem (Commutativity and associativity of $\oplus$)

Let A, B, C be three sets considered over the universal set U, then

1. $A \oplus B = B \oplus A$.
2. $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Exercise 3: Set Operations

Exercise

Given a universal set $S = \{x \in \mathbb{N}_0 : x \leq 10\}$, the set $A = \{x \in S : x \text{ is even}\}$, the set $B = \{x \in S : x < 7\}$, and the set $C = \{x \in S : x > 3\}$. Determine:

1. $A \cup B \cup C$
2. $A \cap B \cap C$
3. $(A \setminus B) \setminus C$
4. $A \setminus (B \setminus C)$
5. $A \oplus B$
6. $(A \oplus B) \oplus C$
7. $A \oplus (B \oplus C)$
8. $A \oplus A^C$

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.
2. Observe that $B \cap C = \{x \in S : (x < 7) \wedge (x > 3)\} = \{x \in S : 3 < x < 7\} = \{4, 5, 6\}$. Therefore $A \cap B \cap C = \{4, 6\}$.

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.
2. Observe that $B \cap C = \{x \in S : (x < 7) \wedge (x > 3)\} = \{x \in S : 3 < x < 7\} = \{4, 5, 6\}$. Therefore $A \cap B \cap C = \{4, 6\}$.
3. $A \setminus B = \{x \in S : (x \in A) \wedge (x \notin B)\} = \{8, 10\}$. Hence $(A \setminus B) \setminus C = \{x \in S : x \in \{8, 10\} \wedge x \notin C\} = \emptyset$.

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.
2. Observe that $B \cap C = \{x \in S : (x < 7) \wedge (x > 3)\} = \{x \in S : 3 < x < 7\} = \{4, 5, 6\}$. Therefore $A \cap B \cap C = \{4, 6\}$.
3. $A \setminus B = \{x \in S : (x \in A) \wedge (x \notin B)\} = \{8, 10\}$. Hence $(A \setminus B) \setminus C = \{x \in S : x \in \{8, 10\} \wedge x \notin C\} = \emptyset$.
4. $B \setminus C = \{x \in S : (x \in B) \wedge (x \notin C)\} = \{0, 1, 2, 3\}$. Hence $A \setminus (B \setminus C) = \{x \in S : x \in A \wedge x \notin \{0, 1, 2, 3\}\} = \{x \in A : x > 3\} = \{x \in S : (x \text{ is even}) \wedge (x > 3)\} = \{4, 6, 8, 10\}$.

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.
2. Observe that $B \cap C = \{x \in S : (x < 7) \wedge (x > 3)\} = \{x \in S : 3 < x < 7\} = \{4, 5, 6\}$. Therefore $A \cap B \cap C = \{4, 6\}$.
3. $A \setminus B = \{x \in S : (x \in A) \wedge (x \notin B)\} = \{8, 10\}$. Hence $(A \setminus B) \setminus C = \{x \in S : x \in \{8, 10\} \wedge x \notin C\} = \emptyset$.
4. $B \setminus C = \{x \in S : (x \in B) \wedge (x \notin C)\} = \{0, 1, 2, 3\}$. Hence $A \setminus (B \setminus C) = \{x \in S : x \in A \wedge x \notin \{0, 1, 2, 3\}\} = \{x \in A : x > 3\} = \{x \in S : (x \text{ is even}) \wedge (x > 3)\} = \{4, 6, 8, 10\}$.
5. $A \oplus B = \{x \in S : x \in A \cup B \text{ and } x \notin A \cap B\} = \{1, 3, 5, 8, 10\}$.

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.
2. Observe that $B \cap C = \{x \in S : (x < 7) \wedge (x > 3)\} = \{x \in S : 3 < x < 7\} = \{4, 5, 6\}$. Therefore $A \cap B \cap C = \{4, 6\}$.
3. $A \setminus B = \{x \in S : (x \in A) \wedge (x \notin B)\} = \{8, 10\}$. Hence $(A \setminus B) \setminus C = \{x \in S : x \in \{8, 10\} \wedge x \notin C\} = \emptyset$.
4. $B \setminus C = \{x \in S : (x \in B) \wedge (x \notin C)\} = \{0, 1, 2, 3\}$. Hence $A \setminus (B \setminus C) = \{x \in S : x \in A \wedge x \notin \{0, 1, 2, 3\}\} = \{x \in A : x > 3\} = \{x \in S : (x \text{ is even}) \wedge (x > 3)\} = \{4, 6, 8, 10\}$.
5. $A \oplus B = \{x \in S : x \in A \cup B \text{ and } x \notin A \cap B\} = \{1, 3, 5, 8, 10\}$.
6. $(A \oplus B) \oplus C = \{x \in S : x \in (A \oplus B) \cup C \text{ and } x \notin (A \oplus B) \cap C\} = \{1, 3, 4, 6, 7, 9\}$.

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.
2. Observe that $B \cap C = \{x \in S : (x < 7) \wedge (x > 3)\} = \{x \in S : 3 < x < 7\} = \{4, 5, 6\}$. Therefore $A \cap B \cap C = \{4, 6\}$.
3. $A \setminus B = \{x \in S : (x \in A) \wedge (x \notin B)\} = \{8, 10\}$. Hence $(A \setminus B) \setminus C = \{x \in S : x \in \{8, 10\} \wedge x \notin C\} = \emptyset$.
4. $B \setminus C = \{x \in S : (x \in B) \wedge (x \notin C)\} = \{0, 1, 2, 3\}$. Hence $A \setminus (B \setminus C) = \{x \in S : x \in A \wedge x \notin \{0, 1, 2, 3\}\} = \{x \in A : x > 3\} = \{x \in S : (x \text{ is even}) \wedge (x > 3)\} = \{4, 6, 8, 10\}$.
5. $A \oplus B = \{x \in S : x \in A \cup B \text{ and } x \notin A \cap B\} = \{1, 3, 5, 8, 10\}$.
6. $(A \oplus B) \oplus C = \{x \in S : x \in (A \oplus B) \cup C \text{ and } x \notin (A \oplus B) \cap C\} = \{1, 3, 4, 6, 7, 9\}$.
7. $B \oplus C = \{x \in S : x \in B \cup C \text{ and } x \notin B \cap C\} = S \setminus \{4, 5, 6\} = \{0, 1, 2, 3, 7, 8, 9, 10\}$. Thus $A \oplus (B \oplus C) = \{x \in S : x \in A \cup (B \oplus C) \text{ and } x \notin A \cap (B \oplus C)\} = \{1, 3, 4, 6, 7, 9\}$.

Solution

1. Observe that $B \cup C = \{x \in S : (x < 7) \vee (x > 3)\} = S$. Therefore $A \cup B \cup C = A \cup S = S$.
2. Observe that $B \cap C = \{x \in S : (x < 7) \wedge (x > 3)\} = \{x \in S : 3 < x < 7\} = \{4, 5, 6\}$. Therefore $A \cap B \cap C = \{4, 6\}$.
3. $A \setminus B = \{x \in S : (x \in A) \wedge (x \notin B)\} = \{8, 10\}$. Hence $(A \setminus B) \setminus C = \{x \in S : x \in \{8, 10\} \wedge x \notin C\} = \emptyset$.
4. $B \setminus C = \{x \in S : (x \in B) \wedge (x \notin C)\} = \{0, 1, 2, 3\}$. Hence $A \setminus (B \setminus C) = \{x \in S : x \in A \wedge x \notin \{0, 1, 2, 3\}\} = \{x \in A : x > 3\} = \{x \in S : (x \text{ is even}) \wedge (x > 3)\} = \{4, 6, 8, 10\}$.
5. $A \oplus B = \{x \in S : x \in A \cup B \text{ and } x \notin A \cap B\} = \{1, 3, 5, 8, 10\}$.
6. $(A \oplus B) \oplus C = \{x \in S : x \in (A \oplus B) \cup C \text{ and } x \notin (A \oplus B) \cap C\} = \{1, 3, 4, 6, 7, 9\}$.
7. $B \oplus C = \{x \in S : x \in B \cup C \text{ and } x \notin B \cap C\} = S \setminus \{4, 5, 6\} = \{0, 1, 2, 3, 7, 8, 9, 10\}$. Thus $A \oplus (B \oplus C) = \{x \in S : x \in A \cup (B \oplus C) \text{ and } x \notin A \cap (B \oplus C)\} = \{1, 3, 4, 6, 7, 9\}$.
8. $A \oplus A^C = \left\{x \in S : x \in A \cup A^C \text{ and } x \notin A \cap A^C\right\} = \{x \in S : x \in S \text{ and } x \notin \emptyset\} = S$.

Generalized Union and Intersection

Due to associative laws, $A \cup (B \cup C)$ [resp. $A \cap (B \cap C)$] and $(A \cup B) \cup C$ [resp. $(A \cap B) \cap C$] can be written as $A \cup B \cup C$ [resp.$A \cap B \cap C$].
In general, the union of $A_1, A_2, \ldots, A_n$ can be expressed as:

$$\bigcup_{i=1}^{n} A_i = \bigcup \{A_i, i = 1, \ldots, n\} = A_1 \cup A_2 \cup \cdots \cup A_n.$$

Moreover, the intersection of $A_1, A_2, \ldots, A_n$ can be expressed as:

$$\bigcap_{i=1}^{n} A_i = \bigcap \{A_i, i = 1, \ldots, n\} = A_1 \cap A_2 \cap \cdots \cap A_n.$$

These generalized union and intersection can be extended for infinitely many sets, the union of $A_1, A_2, \ldots$ is written by $\bigcup\limits_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots$, and the intersection $A_1, A_2, \ldots$ is written by $\bigcap\limits_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots$.

Exercise 4: Generalized Union and Intersection

Exercise

Given the universal set $\mathbb{N}$, and the set $A_i = \{x \mid x \text{ is odd and } x \leq 2i\}$, determine

1. A_1, A_2, A_3, A_4, A_5
2. $\bigcup_{i=1}^{100} A_i$.
3. $\bigcap_{i=1}^{100} A_i$.

Solution:

Exercise 4: Generalized Union and Intersection

Exercise

Given the universal set $\mathbb{N}$, and the set $A_i = \{x \mid x \text{ is odd and } x \leq 2i\}$, determine

1. $A_1,\ A_2,\ A_3,\ A_4,\ A_5$
2. $\bigcup_{i=1}^{100} A_i$.
3. $\bigcap_{i=1}^{100} A_i$.

Solution:

1. $A_1 = \{x \mid x \text{ is odd and } x \leq 2\} = \{1\}$.
   $A_2 = \{x \mid x \text{ is odd and } x \leq 4\} = \{1, 3\}$.
   $A_3 = \{x \mid x \text{ is odd and } x \leq 6\} = \{1, 3, 5\}$.
   $A_4 = \{x \mid x \text{ is odd and } x \leq 8\} = \{1, 3, 5, 7\}$.
   $A_5 = \{x \mid x \text{ is odd and } x \leq 10\} = \{1, 3, 5, 7, 9\}$.

Exercise 4: Generalized Union and Intersection

Exercise

Given the universal set $\mathbb{N}$, and the set $A_i = \{x \mid x \text{ is odd and } x \leq 2i\}$, determine

1. A_1, A_2, A_3, A_4, A_5
2. $\bigcup_{i=1}^{100} A_i$.
3. $\bigcap_{i=1}^{100} A_i$.

Solution:

1. $A_1 = \{x \mid x \text{ is odd and } x \leq 2\} = \{1\}$.
   $A_2 = \{x \mid x \text{ is odd and } x \leq 4\} = \{1, 3\}$.
   $A_3 = \{x \mid x \text{ is odd and } x \leq 6\} = \{1, 3, 5\}$.
   $A_4 = \{x \mid x \text{ is odd and } x \leq 8\} = \{1, 3, 5, 7\}$.
   $A_5 = \{x \mid x \text{ is odd and } x \leq 10\} = \{1, 3, 5, 7, 9\}$.
2. $\bigcup_{i=1}^{100} A_i = \{1\} \cup \{1, 3\} \cup \{1, 3, 5\} \cup \cdots \cup \{1, 3, 5, \ldots, 199\} = \{1, 3, 5, \ldots, 199\} = A_{100}$, because $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_{100}$.

Exercise 4: Generalized Union and Intersection

Exercise

Given the universal set $\mathbb{N}$, and the set $A_i = \{x \mid x \text{ is odd and } x \leq 2i\}$, determine

1. A_1, A_2, A_3, A_4, A_5
2. $\bigcup_{i=1}^{100} A_i$.
3. $\bigcap_{i=1}^{100} A_i$.

Solution:

1. $A_1 = \{x \mid x \text{ is odd and } x \leq 2\} = \{1\}$.
   $A_2 = \{x \mid x \text{ is odd and } x \leq 4\} = \{1, 3\}$.
   $A_3 = \{x \mid x \text{ is odd and } x \leq 6\} = \{1, 3, 5\}$.
   $A_4 = \{x \mid x \text{ is odd and } x \leq 8\} = \{1, 3, 5, 7\}$.
   $A_5 = \{x \mid x \text{ is odd and } x \leq 10\} = \{1, 3, 5, 7, 9\}$.
2. $\bigcup_{i=1}^{100} A_i = \{1\} \cup \{1, 3\} \cup \{1, 3, 5\} \cup \cdots \cup \{1, 3, 5, \ldots, 199\} = \{1, 3, 5, \ldots, 199\} = A_{100}$, because $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_{100}$.
3. $\bigcap_{i=1}^{100} A_i = \{1\} \cap \{1, 3\} \cap \{1, 3, 5\} \cap \cdots \cap \{1, 3, 5, \ldots, 199\} = \{1\} = A_1$, because $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_{100}$.

Generalized Xor

Because $\oplus$ is associative, we can define

$$\bigoplus_{i=1}^{n} A_i = A_1 \oplus A_2 \oplus \cdots \oplus A_n \text{ and}$$
$$\bigoplus_{i=1}^{\infty} A_i = A_1 \oplus A_2 \oplus \cdots$$

Exercise

Given the universal set $\mathbb{N}$, and the set $A_i = \{x \mid x \text{ is odd and } x \leq 2i\}$, determine $\bigoplus_{i=1}^{5} A_i$.

Solution: we have

$$\bigoplus_{i=1}^{5} A_i \quad =$$

Solution: we have

$$\begin{aligned}\bigoplus_{i=1}^{5} A_i &= A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5 \\ &= \end{aligned}$$

Solution: we have

$$
\begin{aligned}
\bigoplus_{i=1}^{5} A_i &= A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5 \\
&= \{1\} \oplus \{1,3\} \oplus \{1,3,5\} \oplus \{1,3,5,7\} \oplus \{1,3,5,7,9\} \\
&= \{3\} \oplus \{1,3,5\} \oplus \{1,3,5,7\} \oplus \{1,3,5,7,9\} \\
&= \{1,5\} \oplus \{1,3,5,7\} \oplus \{1,3,5,7,9\} \\
&= \{3,7\} \oplus \{1,3,5,7,9\} = \{1,5,9\}.
\end{aligned}
$$

Contents

Cartesian Product

The order elements in sets are unimportant. In order to represent a collection of ordered objects, we need different mathematical structure.

Definition (*ordered n-tuple*)

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first component, a_2 as its second component, ..., and a_n as its n-th component. In particular, if $n = 2$, the ordered 2-tuple (a_1, a_2) is called ordered pair. Two ordered n-tuple $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are equal iff $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

Example

We have $(1, 2, 3) \neq (1, 3, 2)$.

Definition

Let A and B be two sets, the Cartesian product of A and B, denoted by $A \times B$, is defined as the set $A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Example

Suppose $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$, then

- $A \times B =$

Example

Suppose $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$, then

- $A \times B = \{(1, 2), (1, 3), (1, 6),$

Example

Suppose $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$, then

- $A \times B = \{(1,2), (1,3), (1,6), (3,2), (3,3), (3,6),$

Example

Suppose $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$, then

- $A \times B = \{(1,2), (1,3), (1,6), (3,2), (3,3), (3,6), (5,2), (5,3), (5,6)\}$.
- $B \times A =$

Example

Suppose $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$, then

- $A \times B = \{(1,2), (1,3), (1,6), (3,2), (3,3), (3,6), (5,2), (5,3), (5,6)\}$.
- $B \times A = \{(2,1), (2,3), (2,5),$

Example

Suppose $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$, then

- $A \times B = \{(1, 2), (1, 3), (1, 6), (3, 2), (3, 3), (3, 6), (5, 2), (5, 3), (5, 6)\}$.
- $B \times A = \{(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5),$

Example

Suppose $A = \{1, 3, 5\}$ and $B = \{2, 3, 6\}$, then

- $A \times B = \{(1,2), (1,3), (1,6), (3,2), (3,3), (3,6), (5,2), (5,3), (5,6)\}$.
- $B \times A = \{(2,1), (2,3), (2,5), (3,1), (3,3), (3,5), (6,1), (6,3), (6,5)\}$.

Observe that $A \times B \neq B \times A$.

Definition

Suppose $A_1, A_2, \ldots, A_n$ are n sets, the Cartesian product of A_1, A_2, $\ldots$, and A_n, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is defined as the set

$$A_1 \times A_2 \times \cdots \times A_n := \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for all } i = 1, 2, \ldots, n\}.$$

If $A_1 = A_2 = \cdots = A_n = A$, then $A \times A \times \cdots \times A$ is written by A^n.
Sometimes we write $\times_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n$.

Contents

Inclusion-Exclusion Principle for Two Sets

Theorem

If A and B are two finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$\begin{aligned} |A \oplus B| &= |A \cup B| - |A \cap B| \\ &= |A| + |B| - |A \cap B| - |A \cap B| \\ &= |A| + |B| - 2\,|A \cap B|\,. \end{aligned}$$

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

We have $|C| =$

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

We have $|C| = 97$, $|M| =$

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

We have $|C| = 97$, $|M| = 68$, and $|C \cap M| =$

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

We have $|C| = 97$, $|M| = 68$, and $|C \cap M| = 12$. So, the number of students that are studying Calculus or Mathematical Logic in the building is

$$|C \cup M| =$$

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

We have $|C| = 97$, $|M| = 68$, and $|C \cap M| = 12$. So, the number of students that are studying Calculus or Mathematical Logic in the building is

$$\begin{aligned} |C \cup M| &= |C| + |M| - |C \cap M| \\ &= \end{aligned}$$

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

We have $|C| = 97$, $|M| = 68$, and $|C \cap M| = 12$. So, the number of students that are studying Calculus or Mathematical Logic in the building is

$$\begin{aligned} |C \cup M| &= |C| + |M| - |C \cap M| \\ &= 97 + 68 - 12 = 153. \end{aligned}$$

Therefore, the number of students that are neither studying Calculus nor Mathematical Logic is

Exercise

Suppose there are 1467 students in a building, 97 students are studying Calculus, 68 students are studying Mathematical Logic, and 12 students studying both Calculus and Mathematical Logic. How many students in that building that are neither studying Calculus nor Mathematical Logic?

Solution:
Suppose

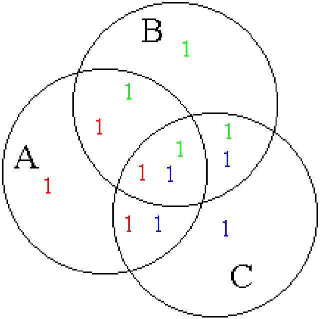
$$\begin{aligned} C &= \{x : x \text{ is a student who is studying Calculus}\}, \\ M &= \{x : x \text{ is a student who is studying Mathematical Logic}\}. \end{aligned}$$

We have $|C| = 97$, $|M| = 68$, and $|C \cap M| = 12$. So, the number of students that are studying Calculus or Mathematical Logic in the building is

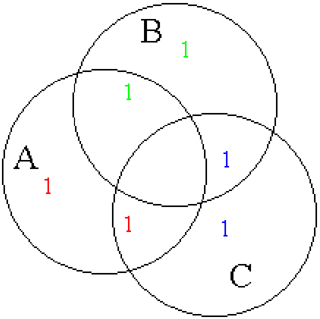
$$\begin{aligned} |C \cup M| &= |C| + |M| - |C \cap M| \\ &= 97 + 68 - 12 = 153. \end{aligned}$$

Therefore, the number of students that are neither studying Calculus nor Mathematical Logic is $1467 - 153 = 1314$ students.

Inclusion-Exclusion for Three Sets



1 red: the region considered when $|A|$ is counted, 1 green: the region considered when $|B|$ is counted, 1 blue: the region considered when $|C|$ is counted. In this illustration, the intersections are counted more than once.



Subtracting with $|A \cap B|$: two 1 red is taken, subtracting with $|A \cap C|$: two 1 blue is taken, subtracting with $|B \cap C|$: two 1 green is taken. The calculation is almost correct, except for the intersection of the three sets. Hence, we need to add $|A \cap B \cap C|$.

Consequently, we have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - (|A \cap B| + |A \cap C| + |B \cap C|) \\ &\quad + (|A \cap B \cap C|) \,. \end{aligned}$$

Exercise

In a building there are 115 students learning Mathematical Logic, 71 students learning Calculus, and 56 students learning Algorithm. Among these students, 25 people learning both Mathematical Logic and Calculus, 14 people learning both Mathematical Logic and Algorithm, and 9 people learning both Calculus and Algorithm. If there are 196 students learning at least one of Mathematical Logic, Calculus, or Algorithm, how many students are learning all three subjects?

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| = 14$, and $|C \cap A| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| = 14$, and $|C \cap A| = 9$. Since there are 196 students that are learning at least one of these three subjects, we have $|M \cup C \cup A| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| = 14$, and $|C \cap A| = 9$. Since there are 196 students that are learning at least one of these three subjects, we have $|M \cup C \cup A| = 196$. By inclusion-exclusion principle, we have

$$|M \cup C \cup A| =$$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| = 14$, and $|C \cap A| = 9$. Since there are 196 students that are learning at least one of these three subjects, we have $|M \cup C \cup A| = 196$. By inclusion-exclusion principle, we have

$$|M \cup C \cup A| = |M| + |C| + |A|$$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| = 14$, and $|C \cap A| = 9$. Since there are 196 students that are learning at least one of these three subjects, we have $|M \cup C \cup A| = 196$. By inclusion-exclusion principle, we have

$$\begin{aligned} |M \cup C \cup A| &= |M| + |C| + |A| \\ &\quad - (|M \cap C| + |M \cap A| + |C \cap A|) \end{aligned}$$

Solution: Suppose

$$
\begin{aligned}
M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\
C &= \{x : x \text{ is a student who is learning Calculus}\}, \\
A &= \{x : x \text{ is a student who is learning Algorithm}\}.
\end{aligned}
$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| = 14$, and $|C \cap A| = 9$. Since there are 196 students that are learning at least one of these three subjects, we have $|M \cup C \cup A| = 196$. By inclusion-exclusion principle, we have

$$
\begin{aligned}
|M \cup C \cup A| &= |M| + |C| + |A| \\
&\quad - (|M \cap C| + |M \cap A| + |C \cap A|) \\
&\quad + |M \cap C \cap A| \\
196 &= 115 + 71 + 56 - (25 + 14 + 9) + |M \cap C \cap A|.
\end{aligned}
$$

Therefore $|M \cap C \cap A| =$

Solution: Suppose

$$\begin{aligned} M &= \{x : x \text{ is a student who is learning Mathematical Logic}\}, \\ C &= \{x : x \text{ is a student who is learning Calculus}\}, \\ A &= \{x : x \text{ is a student who is learning Algorithm}\}. \end{aligned}$$

We have $|M| = 115$, $|C| = 71$, $|A| = 56$, $|M \cap C| = 25$, $|M \cap A| = 14$, and $|C \cap A| = 9$. Since there are 196 students that are learning at least one of these three subjects, we have $|M \cup C \cup A| = 196$. By inclusion-exclusion principle, we have

$$\begin{aligned} |M \cup C \cup A| &= |M| + |C| + |A| \\ &\quad - (|M \cap C| + |M \cap A| + |C \cap A|) \\ &\quad + |M \cap C \cap A| \\ 196 &= 115 + 71 + 56 - (25 + 14 + 9) + |M \cap C \cap A|. \end{aligned}$$

Therefore $|M \cap C \cap A| = 2$.

Contents

Set Partition

Definition

Let A be a set. The collection of subsets

$$A_1 \subseteq A,\ A_2 \subseteq A, \ldots,\ A_n \subseteq A$$

is called a **partition** of set A if:

1. $A_i \neq \emptyset$, for all $i = 1, \ldots, n$
2. $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i = A$, and
3. $A_i \cap A_j = \emptyset$ for all different i and j.

Example

Suppose $A = \{1, 2, 3 \ldots, 10\}$. The collection $\{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}, \{9, 10\}\}$ and $\{\{1\}, \{2, 3, 5, 7\}, \{4, 6, 8, 10\}, \{9\}\}$

Example

Suppose $A = \{1, 2, 3 \ldots, 10\}$. The collection $\{\{1,2,3\}, \{4,5\}, \{6,7,8\}, \{9,10\}\}$ and $\{\{1\}, \{2,3,5,7\}, \{4,6,8,10\}, \{9\}\}$ are partitions of A. However, the collection $\{\{1,2,3,4\}, \{4,5,6\}, \{6,7,8\}, \{8,9,10\}\}$ and $\{\{1,2,3\}, \{4,5,6\}, \{7,8\}\}$

Example

Suppose $A = \{1, 2, 3 \ldots, 10\}$. The collection $\{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}, \{9, 10\}\}$ and $\{\{1\}, \{2, 3, 5, 7\}, \{4, 6, 8, 10\}, \{9\}\}$ are partitions of A. However, the collection $\{\{1, 2, 3, 4\}, \{4, 5, 6\}, \{6, 7, 8\}, \{8, 9, 10\}\}$ and $\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}\}$ are not partition of A.

Remark

For infinite sets, the number of subsets in its partition is also infinite. In this case, we change the expression $\bigcup_{i=1}^{n} A$ with $\bigcup_{i=1}^{\infty} A_i$.

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution:

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution: (1) yes,

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution: (1) yes, (2) no, because $\mathbf{k}$ is missing,

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution: (1) yes, (2) no, because $\mathbf{k}$ is missing, (3) no, because $\mathbf{t} \notin S$,

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution: (1) yes, (2) no, because $\mathbf{k}$ is missing, (3) no, because $\mathbf{t} \notin S$, (4) yes,

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution: (1) yes, (2) no, because $\mathbf{k}$ is missing, (3) no, because $\mathbf{t} \notin S$, (4) yes, (5) yes ($\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\} = \{\mathbf{b}, \mathbf{o}, \mathbf{k}\}$),

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution: (1) yes, (2) no, because $\mathbf{k}$ is missing, (3) no, because $\mathbf{t} \notin S$, (4) yes, (5) yes ($\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\} = \{\mathbf{b}, \mathbf{o}, \mathbf{k}\}$), (6) no, because $\emptyset$ is not allowed,

Exercise: Partition

Exercise

Suppose $S = \{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{k}, \mathbf{c}, \mathbf{s}\}$. Determine whether each of these collections is a partition of S.

1. $\{\{\mathbf{m}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{b}, \mathbf{s}\}\}$
2. $\{\{\mathbf{c}, \mathbf{o}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{u}, \mathbf{s}\}, \{\mathbf{r}\}\}$
3. $\{\{\mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}\}, \{\mathbf{m}, \mathbf{u}, \mathbf{s}, \mathbf{t}\}\}$
4. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}, \mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}\}$
5. $\{\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\}, \{\mathbf{r}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{s}\}\}$
6. $\{\{\mathbf{u}, \mathbf{m}, \mathbf{b}\}, \{\mathbf{r}, \mathbf{o}, \mathbf{c}, \mathbf{k}, \mathbf{s}\}, \emptyset\}$
7. $\{\{\mathbf{b}, \mathbf{u}, \mathbf{m}\}, \{\mathbf{c}, \mathbf{o}, \mathbf{r}, \mathbf{k}, \mathbf{s}\}\}$

Solution: (1) yes, (2) no, because $\mathbf{k}$ is missing, (3) no, because $\mathbf{t} \notin S$, (4) yes, (5) yes ($\{\mathbf{b}, \mathbf{o}, \mathbf{o}, \mathbf{k}\} = \{\mathbf{b}, \mathbf{o}, \mathbf{k}\}$), (6) no, because $\emptyset$ is not allowed, (7), yes.

Contents

Mathematical Proofs Concerning Set Theory

The proofs of statements involving set theory are obtained using the definition for set equality, subset, and elementary set operations. Recall that, from proof methods discussion, a mathematical proof can be: a direct proof, an indirect proof by contrapositive, or an indirect proof by contradiction.

Theorem

Let A and B be two sets. If $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, then $A \subseteq C$.

Proof

Theorem

Let A and B be two sets. If $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, then $A \subseteq C$.

Proof

With the assumption that $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, we must prove that $A \subseteq C$, meaning that for all $x \in A$ then $x \in C$.

Theorem

Let A and B be two sets. If $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, then $A \subseteq C$.

Proof

With the assumption that $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, we must prove that $A \subseteq C$, meaning that for all $x \in A$ then $x \in C$.
Suppose $x \in A$, because $A \subseteq (B \cup C)$, then $x \in B \cup C$. This means $x \in B$ or $x \in C$.

Theorem

Let A and B be two sets. If $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, then $A \subseteq C$.

Proof

With the assumption that $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, we must prove that $A \subseteq C$, meaning that for all $x \in A$ then $x \in C$.

Suppose $x \in A$, because $A \subseteq (B \cup C)$, then $x \in B \cup C$. This means $x \in B$ or $x \in C$.

Because $A \cap B = \emptyset$, it is impossible that x satisfies both $x \in A$ and $x \in B$. Since $x \in A$ is already true, we must have $x \notin B$.

Theorem

Let A and B be two sets. If $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, then $A \subseteq C$.

Proof

With the assumption that $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, we must prove that $A \subseteq C$, meaning that for all $x \in A$ then $x \in C$.
Suppose $x \in A$, because $A \subseteq (B \cup C)$, then $x \in B \cup C$. This means $x \in B$ or $x \in C$.
Because $A \cap B = \emptyset$, it is impossible that x satisfies both $x \in A$ and $x \in B$. Since $x \in A$ is already true, we must have $x \notin B$.
Therefore, since $x \in B$ or $x \in C$, and also $x \notin B$ from previous argument, we must have $x \in C$.

Theorem

Let A and B be two sets. If $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, then $A \subseteq C$.

Proof

With the assumption that $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, we must prove that $A \subseteq C$, meaning that for all $x \in A$ then $x \in C$.
Suppose $x \in A$, because $A \subseteq (B \cup C)$, then $x \in B \cup C$. This means $x \in B$ or $x \in C$.
Because $A \cap B = \emptyset$, it is impossible that x satisfies both $x \in A$ and $x \in B$. Since $x \in A$ is already true, we must have $x \notin B$.
Therefore, since $x \in B$ or $x \in C$, and also $x \notin B$ from previous argument, we must have $x \in C$.
So, if $x \in A$ with $A \cap B = \emptyset$ and $A \subseteq (B \cup C)$, then $x \in C$. Therefore $A \subseteq C$. □