Elementary Set Theory Part 1: Notation, Set Equality, Cardinality, and Power Set

Mathematical Logic – First Term 2023-2024

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Acknowledgements

This slide is compiled using the materials in the following sources:

1. *Discrete Mathematics and Its Applications* (Chapter 2), 8th Edition, 2019, by K. H. Rosen (primary reference).
2. *Discrete Mathematics with Applications* (Chapter 6), 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Discrete Mathematics 1 (2012) slides at Fasilkom UI by B. H. Widjaja.
5. Discrete Mathematics 1 (2010) slides at Fasilkom UI by A. A. Krisnadhi.

Some figures are excerpted from those sources. This slide is intended for internal academic purpose in SoC Telkom University. No slides are ever free from error nor incapable of being improved. Please convey your comments and corrections (if any) to `<pleasedontspam>@telkomuniversity.ac.id`.

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Set Definition and Notation

Set is an important mathematical object in modern mathematics and computer science.

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A set is an unordered collection of distinct objects. These objects are called elements or members of the set. A set is said to contain its elements.

From this definition:

- Elements duplication does not matter.
- The order of appearance for the elements does not matter.

Set Notation

- Sets are usually denoted with uppercase letters: $A, B, C, \ldots, X, Y, Z$, or with indices if necessary, such as: $A_1, A_2, \ldots, X_1, X_2, \ldots$.
- Elements of sets are usually denoted with lowercase letters: $a, b, c, \ldots x, y, z$, or with indices if necessary, such as: $a_1, a_2, \ldots, x_1, x_2, \ldots$.
- The notation $x \in A$ means x is an element (member) of A, or equivalently A contains x.
- The notation $x \notin A$ means x **is not** an element (member) of A, or equivalently A **does not contain** x.
- The notation $\emptyset$ or $\varnothing$ or $\{\}$ denotes the empty set/ null set, that is, the set that has no elements.
  The proposition $x \in \emptyset$ is always F and the proposition $x \notin \emptyset$ is always T.

Simple Examples

Example

Suppose

- $A = \{loki, thor, odin, freia\}$,
- $B = \{finn, 10, \sqrt[3]{2}, rey\}$,
- $C = \{9, \{9\}, \{\{9\}\}\}$,
- $D = \{Jon\ Snow, \{Sansa\ Stark, Tyrion\ Lannister\}\}$.

We have:

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- $loki \in A$, $odin \in A$, $dormammu \notin A$,

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We have:

- $loki \in A$, $odin \in A$, $dormammu \notin A$,
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- $9 \in C$, $\{9\} \in C$, $\{\{9\}\} \in C$, $\{\{\{9\}\}\} \notin C$,

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We have:

- $loki \in A$, $odin \in A$, $dormammu \notin A$,
- $finn \in B$, $han \notin B$, $luke \notin B$,
- $9 \in C$, $\{9\} \in C$, $\{\{9\}\} \in C$, $\{\{\{9\}\}\} \notin C$,
- $Jon\ Snow \in D$, $Sansa\ Stark \notin D$,
  $Tyrion\ Lannister \in \{Sansa\ Stark, Tyrion\ Lannister\}$.

Example

We have:

- the set of the first four positive prime numbers:

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We have:

- the set of the first four positive prime numbers: $\{2, 3, 5, 7\}$,
- the set of the first five positive odd numbers:

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- the set of the first four positive prime numbers: $\{2, 3, 5, 7\}$,
- the set of the first five positive odd numbers: $\{1, 3, 5, 7, 9\}$,
- the set of the first 100 positive even numbers:

Example

We have:

- the set of the first four positive prime numbers: $\{2, 3, 5, 7\}$,
- the set of the first five positive odd numbers: $\{1, 3, 5, 7, 9\}$,
- the set of the first 100 positive even numbers: $\{2, 4, 6, \ldots, 200\}$,
- the set of integers:

Example

We have:

- the set of the first four positive prime numbers: $\{2, 3, 5, 7\}$,
- the set of the first five positive odd numbers: $\{1, 3, 5, 7, 9\}$,
- the set of the first 100 positive even numbers: $\{2, 4, 6, \ldots, 200\}$,
- the set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Example

Suppose $A_1 = \{a, b\}$, $A_2 = \{a, \{a, b\}\}$, $A_3 = \{b, \{a, \{a, b\}\}\}$. We have:

1. A_1 contains two elements, i.e.,

Example

Suppose $A_1 = \{a, b\}$, $A_2 = \{a, \{a, b\}\}$, $A_3 = \{b, \{a, \{a, b\}\}\}$. We have:

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Example

Suppose $A_1 = \{a, b\}$, $A_2 = \{a, \{a, b\}\}$, $A_3 = \{b, \{a, \{a, b\}\}\}$. We have:

1. A_1 contains two elements, i.e., a and b,
2. A_2 contains two elements, i.e., a and $\{a, b\}$,
3. A_3 contains two elements, i.e.,

Example

Suppose $A_1 = \{a, b\}$, $A_2 = \{a, \{a, b\}\}$, $A_3 = \{b, \{a, \{a, b\}\}\}$. We have:

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2. A_2 contains two elements, i.e., a and $\{a, b\}$,
3. A_3 contains two elements, i.e., b and $\{a, \{a, b\}\}$.

Therefore:

Example

Suppose $A_1 = \{a, b\}$, $A_2 = \{a, \{a, b\}\}$, $A_3 = \{b, \{a, \{a, b\}\}\}$. We have:

1. A_1 contains two elements, i.e., a and b,
2. A_2 contains two elements, i.e., a and $\{a, b\}$,
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Therefore:

1. $a \in A_1$, $b \in A_1$,

Example

Suppose $A_1 = \{a, b\}$, $A_2 = \{a, \{a, b\}\}$, $A_3 = \{b, \{a, \{a, b\}\}\}$. We have:

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2. A_2 contains two elements, i.e., a and $\{a, b\}$,
3. A_3 contains two elements, i.e., b and $\{a, \{a, b\}\}$.

Therefore:

1. $a \in A_1$, $b \in A_1$,
2. $a \in A_2$, $A_1 \in A_2$, $b \notin A_2$,

Example

Suppose $A_1 = \{a, b\}$, $A_2 = \{a, \{a, b\}\}$, $A_3 = \{b, \{a, \{a, b\}\}\}$. We have:

1. A_1 contains two elements, i.e., a and b,
2. A_2 contains two elements, i.e., a and $\{a, b\}$,
3. A_3 contains two elements, i.e., b and $\{a, \{a, b\}\}$.

Therefore:

1. $a \in A_1$, $b \in A_1$,
2. $a \in A_2$, $A_1 \in A_2$, $b \notin A_2$,
3. $b \in A_3$, $A_2 \in A_3$, $a \notin A_3$, $A_1 \notin A_3$.

This example illustrates that a set can be an element of another set. In other words: a member of a particular set can be a set.

Defining and Writing a Set

A set can be represented in the following ways.

1. using a list (*roster method*):
   1. $A = \{x_1, x_2, \ldots, x_n\}$ for a set with finitely many elements;
   2. $A = \{x_1, x_2, \ldots\}$ for a set with infinitely many elements.

   The ellipses symbol "$\ldots$" are used when the general pattern of the elements is obvious.
2. using *set builder notation* with a particular predicate:
   1. $A = \{x \mid P(x)\}$ or $A = \{x : P(x)\}$
   2. $A = \{x \in U \mid P(x)\}$ or $A = \{x \in U : P(x)\}$, in this notation, U is another set in the universe of discourse which restrict the elements in the set A
   3. Sometimes the set U is a universal set or a universe of discourse for the discussion.

   The notation $A = \{x \in U \mid P(x)\}$ is read: A contains all elements x in U such that $P(x)$.
   $P(x)$ is a unary predicate with variable x.

Examples of Set Builder Notation

Example

Suppose:

1. $A = \{w \mid w \text{ is a positive integer less than } 10\}$,
2. $B = \{x \mid P(x)\}$ where $P(x) : x$ is an odd number between 20 and 30,
3. $C = \{y \mid P(y)\}$ where $P(y) : y$ is a positive factor of 10,
4. $D = \{z \mid z \text{ is a positive prime factor of } 12\}$.

Then:

1. $A =$

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Then:

1. $A = \{1, 2, 3, \ldots, 8, 9\}$,
2. $B =$

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Then:

1. $A = \{1, 2, 3, \ldots, 8, 9\}$,
2. $B = \{21, 23, 25, 27, 29\}$,
3. $C =$

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Then:

1. $A = \{1, 2, 3, \ldots, 8, 9\}$,
2. $B = \{21, 23, 25, 27, 29\}$,
3. $C = \{1, 2, 5, 10\}$,
4. $D = \{2, 3\}$.

Contents

Important Sets of Numbers

The set of natural numbers: this set is denoted by $\mathbb{N}$, $\mathbf{N}$, or $\mathcal{N}$. In this course $\mathbb{N} = \{1, 2, 3, \ldots\}$. However, many computer science references define $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

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The set of counting numbers: this set is denoted by $\mathbb{N}_0$, $\mathbf{N}_0$, or $\mathcal{N}_0$. In this course $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

- Every natural number is also a counting number.

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- Every natural number is also a counting number.

The set of integers: this set is denoted by $\mathbb{Z}$, $\mathbf{Z}$, or $\mathcal{Z}$, , and it is defined as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

- Every counting number is also an integer.
- The set of positive integers is denoted by $\mathbb{Z}^+$ or $\mathbb{Z}_{>0}$, we have $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Thus, the set of positive integers is equal to the set of natural numbers, in other words, $\mathbb{Z}^+ = \mathbb{N}$.

The set of rational numbers: this set is denoted by $\mathbb{Q}$, $\mathbf{Q}$, or $\mathcal{Q}$, and it is defined as $\mathbb{Q} := \left\{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$.

- Since there is no elements duplication in a set, $\mathbb{Q}$ can also be defined as $\mathbb{Q} := \left\{\frac{a}{b} \mid a \in \mathbb{Z},\, b \in \mathbb{N}, \text{ and } \gcd(a, b) = 1\right\}$.
- Every integer m can be expressed as $\frac{m}{1}$. Therefore every integer is also a rational number.
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The set of real numbers: this set is denoted by $\mathbb{R}$, $\mathbf{R}$, or $\mathcal{R}$. This set encompasses all numbers that can be "measured (continuously) along any geometric line".

- The set of real numbers includes the set of rational numbers ($\mathbb{Q}$) and the set of irrational numbers.
- The set of positive real numbers is denoted by $\mathbb{R}^+$ or $\mathbb{R}_{>0}$.

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- The set of positive real numbers is denoted by $\mathbb{R}^+$ or $\mathbb{R}_{>0}$.

The set of complex numbers: this set is denoted by $\mathbb{C}$, $\mathbf{C}$, or $\mathcal{C}$, and it is defined as $\mathbb{C} := \left\{ a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1 \right\}$.

- Every real number a can be expressed as $a + 0i$. Therefore every real number is also a complex number.

Contents

Universal Set

Universal Set

A universal set (or universe of discourse) is a set that contains all the objects under consideration. This set is usually denoted by U. A universal set is not unique and highly depends on the objects under consideration.

Example

1. When we use the universal set $U = \{x \mid (x \in \mathbb{N}) \wedge (x \leq 100)\}$, our objects of interest are only natural numbers which are not greater than 100. We do not consider other numbers outside U, such as -1, 101, or $\frac{1}{2}$.
2. When we use the universal set $U = \mathbb{R}$, our objects of interest are only real numbers. We do not consider other elements outside U, such as $\sqrt{-2}$ and $1 + \sqrt{-3}$.

Venn Diagram

Venn Diagram

Venn diagram is used to graphically represent the relationship of one or several sets with respect to a particular universal set. This universal set is depicted by a rectangle.

Suppose we have the universal set
$U = \{\alpha \mid \alpha$ is one of the 26 letters in the standard English alphabet$\}$ and
$V = \{\beta \mid \beta$ is one of the vowels in the standard English alphabet$\}$. The Venn diagram representation for the relationship of U and V is

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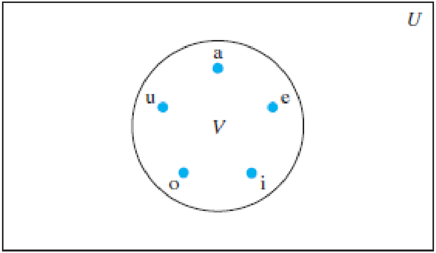

Suppose we have the universal set $U = \{x \mid (x \in \mathbb{N}) \wedge (x \leq 8)\}$, $A = \{1, 2, 3, 5\}$, and $B = \{2, 5, 6, 8\}$. The Venn diagram representation for the relationship of U, A, and B is

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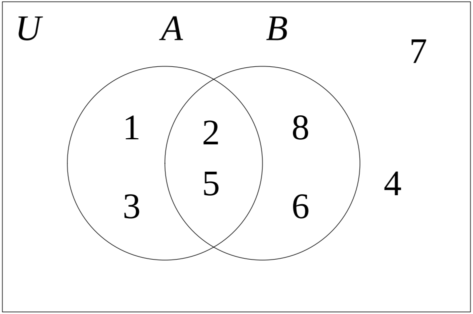

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Set Equality

Definition (Set Equality)

Two sets A and B are equal, denoted by $A = B$, if A and B contain the same elements. Otherwise, A and B are not equal and denoted by $A \neq B$.

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$A = B$ if & only if (iff) the predicate formula $\forall x\,(x \in A \leftrightarrow x \in B)$ is true.

Example

Suppose we have:

- $A = \{1, 2, 3, 4, 6, 12\}$,
- $B = \{x \mid x \text{ is a positive factor of } 12\}$,
- $C = \{1, 2, 3\}$,
- $D = \{1, 2, 2, 3, 3, 3\}$.

Then we have:

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Suppose we have:

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- $B = \{x \mid x$ is a positive factor of $12\}$,
- $C = \{1, 2, 3\}$,
- $D = \{1, 2, 2, 3, 3, 3\}$.

Then we have: $A = B$, $C = D$, $A \neq C$, $A \neq D$, $B \neq C$, $B \neq D$.

Subset

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The set A is a subset of B, denoted by $A \subseteq B$, if every element of A is also an element of B. Furthermore, in this case, we say that the set be B is a superset of A and it is denoted by $B \supseteq A$.

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Definition (Proper Subset)

The set A is a proper subset of B, denoted by $A \subset B$ or $A \subsetneq B$, if $A \subseteq B$ but $A \neq B$.

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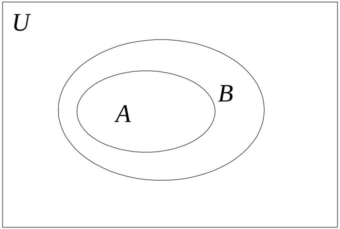
Definition (Proper Subset)

The set A is a proper subset of B, denoted by $A \subset B$ or $A \subsetneq B$, if $A \subseteq B$ but $A \neq B$.

$A \subset B$ iff the predicate formula $\forall x\,(x \in A \to x \in B) \wedge \exists x\,(x \in B \wedge x \notin A)$ is true.

Venn Diagram for the condition $A \subset B$ can be depicted as follows.



Example

Suppose:

- $A = \{1, 2, 3, 4, 5\}$,
- $B = \{1, 2, 3, 4\}$,
- $C = \{x \mid (x \in \mathbb{N}) \land (x + 5 < 10)\}$,
- $\emptyset = \{\}$.

Then we have:

Example

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- $A = \{1, 2, 3, 4, 5\}$,
- $B = \{1, 2, 3, 4\}$,
- $C = \{x \mid (x \in \mathbb{N}) \wedge (x + 5 < 10)\}$,
- $\emptyset = \{\}$.

Then we have:

1. $\emptyset \subseteq A$, $\emptyset \subseteq B$, $\emptyset \subseteq C$, and $\emptyset \subseteq \emptyset$, and $\emptyset \subset A$, $\emptyset \subset B$, $\emptyset \subset C$, but **it is not true that** $\emptyset \subset \emptyset$,

Example

Suppose:

- $A = \{1, 2, 3, 4, 5\}$,
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Then we have:

1. $\emptyset \subseteq A$, $\emptyset \subseteq B$, $\emptyset \subseteq C$, and $\emptyset \subseteq \emptyset$, and $\emptyset \subset A$, $\emptyset \subset B$, $\emptyset \subset C$, but **it is not true that** $\emptyset \subset \emptyset$,
2. $B \subseteq A$ and $B \subset A$, because $5 \in A$ but $5 \notin B$,

Example

Suppose:

- $A = \{1, 2, 3, 4, 5\}$,
- $B = \{1, 2, 3, 4\}$,
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Then we have:

1. $\emptyset \subseteq A$, $\emptyset \subseteq B$, $\emptyset \subseteq C$, and $\emptyset \subseteq \emptyset$, and $\emptyset \subset A$, $\emptyset \subset B$, $\emptyset \subset C$, but **it is not true that** $\emptyset \subset \emptyset$,
2. $B \subseteq A$ and $B \subset A$, because $5 \in A$ but $5 \notin B$,
3. $C \subseteq A$, $C \subset A$ (because $5 \in A$ but $5 \notin C$), $C \subseteq B$, and $B \subseteq C$ (because $B = C$).

Example

For the prominent set of numbers, we have $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

More about set equality and subset

Theorem

For any set A:

1. $\emptyset \subseteq A$,
2. $A \subseteq A$.

Proof

The proof is left as an exercise for the reader.

Theorem

If A and B are two sets, then $A = B$ if & only if $A \subseteq B$ **and** $A \supseteq B$.

Proof

The proof is left as an exercise for the reader.

Theorem

For any set A, B, and C: if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof

The proof is left as an exercise for the reader.

Exercise 1: Set Equality & Subset

Exercise

For each of these statements, choose T if the statement is true, or F if the statement is false.

1. $\{1, 3, 5\} = \{3, 5, 1\}$ T F
2. $\{1, 3, 5\} = \{1, 3, 3, 3, 5, 5, 5, 5, 5\}$ T F
3. $\{1, \{1\}, \{\{1\}\}\} = \{1\}$ T F
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Solution:
No. 1: T, because order of elements is irrelevant. No. 2: T, because elements duplication is irrelevant. No. 3: F, because the set $\{1,\{1\},\{\{1\}\}\}$ contains three elements, i.e., 1, $\{1\}$, and $\{\{1\}\}$; whereas $\{1\}$ only contains one element, i.e., 1.

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No. 4: T, because $1 \in \{1\}$ and $1 \in \{1, \{1\}\}$.

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Exercise 2: Relationship Between Two Sets

Exercise

Fill the bank with one of the following symbols: $=$, $\subset$, $\supset$, $\in$, $\ni$, or X if the relation $=, \subset, \supset, \in, \ni$ are not appropriate.

(1)	$\{\emptyset\}$	☐	$\{\{\}\}$
(2)	$\emptyset$	☐	$\{0\}$
(3)	$\{\emptyset, \{\emptyset\}\}$	☐	$\{\emptyset\}$
(4)	$\{1\}$	☐	$\{\{1\}, \{\{1\}\}, \{\{\{1\}\}\}\}$
(5)	$\{\{\}\}$	☐	$\{\}$
(6)	$\{\emptyset\}$	☐	$\{0\}$

Solution to Exercise 2

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4. $\{1\} \in \{\{1\}, \{\{1\}\}, \{\{\{1\}\}\}\}$.
5. Since $\{\{\}\} \ni \{\}$ and $\{\{\}\} \supset \{\}$, then we have two correct answers, i.e., $\ni$ and $\supset$.
6. $\{\emptyset\}$ is a set that contains exactly one member; an empty set. The set $\{0\}$ is also a set that contains exactly one member; a number 0. Since it is obvious that $\{\emptyset\} \neq \{0\}$, there is no appropriate symbol among $=, \subset, \supset, \in, \ni$ for the relationship between $\{\emptyset\}$ and $\{0\}$.

Contents

Cardinality of (Finite) Sets

Definition

Let A be a set:

- A is finite iff A contains exactly n elements, for some nonnegative integer n;
- in this case, n is the cardinality of A, and it is denoted by $|A|$, $n(A)$, or $\#A$,
- A is infinite iff A is not finite.

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- If $A = \{m \in \mathbb{N} \mid m < 10 \text{ and } m \text{ is odd}\}$, then $|A| =$

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- If $A = \{m \in \mathbb{N} \mid m < 10 \text{ and } m \text{ is odd}\}$, then $|A| = 5$.
- $|\emptyset| = 0$, $|\{\emptyset\}| = 1$, $|\{\{\emptyset\}\}| = 1$, $|\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}| = 3$.
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are examples of infinite sets.

Set Equivalence

Definition

Two sets A and B are equivalent, denoted by $A \sim B$, if they have same cardinality. If A and B are finite, then $A \sim B$ if $|A| = |B|$.

Example

The sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$ satisfy $A \neq B$ but $A \sim B$ because $|A| = |B| = 4$.

Power Set

Definition

Let A be a set. The power set of A is the set of all subsets of A. The power set of A is denoted by 2^A, $\mathcal{P}(A)$, or $\wp(A)$.

Example

- $\mathcal{P}(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.
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Theorem

If A is a set and $|A| = n$, then $|\mathcal{P}(A)| = \left|2^A\right| = 2^n$.

Proof

The proof can be derived using mathematical induction and is left for the reader as an exercise.