Mathematical Induction
Part 2: Strong Induction

Mathematical Logic – First Term 2022-2023

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Acknowledgements

This slide is compiled using the materials in the following sources:

1. *Discrete Mathematics and Its Applications* (Chapter 5), 8th Edition, 2019, by K. H. Rosen (primary reference).
2. *Discrete Mathematics with Applications* (Chapter 5), 5th Edition, 2018, by S. S. Epp.
3. Discrete Mathematics 1 (2012) slides at Fasilkom UI by B. H. Widjaja.
4. Discrete Mathematics 1 (2010) slides at Fasilkom UI by A. A. Krisnadhi.

Some figures are excerpted from those sources. This slide is intended for internal academic purpose in SoC Telkom University. No slides are ever free from error nor incapable of being improved. Please convey your comments and corrections (if any) to `<pleasedontspam>@telkomuniversity.ac.id`.

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Strong Induction: Motivation

The (ordinary) mathematical induction cannot always be used to prove mathematical statements of the form $\forall n P(n)$.

Theorem

Let a_n be a sequence defined recursively as: $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n \geq 4$. The value of a_n satisfies $a_n < 2^n$ for all $n \in \mathbb{N}$.

According to this recursive definition, we have

$$\begin{aligned} a_1 &= 1 < 2^1,\ a_2 = 2 < 2^2,\ a_3 = 3 < 2^3 \\ a_4 &= \end{aligned}$$

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$$\begin{aligned}
a_1 &= 1 < 2^1,\ a_2 = 2 < 2^2,\ a_3 = 3 < 2^3 \\
a_4 &= a_3 + a_2 + a_1 = 3 + 2 + 1 = 6 < 2^4 \\
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According to this recursive definition, we have

$$\begin{aligned}
a_1 &= 1 < 2^1,\ a_2 = 2 < 2^2,\ a_3 = 3 < 2^3 \\
a_4 &= a_3 + a_2 + a_1 = 3 + 2 + 1 = 6 < 2^4 \\
a_5 &= a_4 + a_3 + a_2 = 6 + 3 + 2 = 11 < 2^5 \\
a_6 &=
\end{aligned}$$

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According to this recursive definition, we have

$$\begin{aligned} a_1 &= 1 < 2^1,\ a_2 = 2 < 2^2,\ a_3 = 3 < 2^3 \\ a_4 &= a_3 + a_2 + a_1 = 3 + 2 + 1 = 6 < 2^4 \\ a_5 &= a_4 + a_3 + a_2 = 6 + 3 + 2 = 11 < 2^5 \\ a_6 &= a_5 + a_4 + a_3 = 11 + 6 + 3 = 20 < 2^6 \end{aligned}$$

We first try to prove that $a_n < 2^n$ for all $n \in \mathbb{N}$ using the (ordinary) mathematical induction.

Proof (?)

Suppose

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Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

Basis step: Observe that $P(1) \equiv a_1 < 2^1$. Since $a_1 = 1$, then obviously $P(1)$ is true.

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Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Inductive step: Suppose $k \in \mathbb{N}$ and assume that $P(k) \equiv a_k < 2^k$ is true . We must show that $P(k+1) \equiv a_{k+1} < 2^{k+1}$ is also true. Observe that

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} + a_{k-2} \\ &< 2^k + a_{k-1} + a_{k-2} \text{ (by inductive hypothesis)} \end{aligned}$$

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Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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What's next ???

- We cannot complete the prove of the theorem using (ordinary) mathematical induction, because we have no information regarding the values of a_{k-1} and a_{k-2}.
- *Strong induction* is a type of mathematical induction that can be used to prove the statements which require "stronger" inductive hypothesis.

Strong Induction: Structure

- How do we prove mathematical statements using strong induction?
- Suppose we have a statement of the form: $\forall n P(n)$, where n is a variable over $\mathbb{N}_0$.

Strong Induction

In order to prove that $\forall n P(n)$ is true, then we have to do the following:

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In order to prove that $\forall n P(n)$ is true, then we have to do the following:

1. Basis step: prove that $P(k)$ is true for several values of k (possibly more than one). Clearly, we need to show that $P(0)$ is true since 0 is the smallest element in $\mathbb{N}_0$.

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Strong Induction

In order to prove that $\forall n P(n)$ is true, then we have to do the following:

1. Basis step: prove that $P(k)$ is true for several values of k (possibly more than one). Clearly, we need to show that $P(0)$ is true since 0 is the smallest element in $\mathbb{N}_0$.
2. Inductive step: prove that, for any integer i such that $0 \leq i \leq k$, if $P(i)$ is true, then $P(k+1)$ is also true. In other words, we need to show that

$$(P(0) \wedge P(1) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)$$

is true.

- Just like the ordinary induction, the basis step in strong induction may not be started from 0.

Contents

Proofs Using Strong Induction

Theorem (Theorem 4)

Let a_n be a sequence defined recursively as: $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n \geq 4$. The value of a_n satisfies $a_n < 2^n$ for all $n \in \mathbb{N}$.

Proof (Proof of Theorem 4)

Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

Basis step:

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Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.
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Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Proof (Proof of Theorem 4)

Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Therefore $P(k+1)$ is also true.

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Suppose $P(n) \equiv a_n < 2^n$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Therefore $P(k+1)$ is also true.

We have completed the basis step and the inductive step. Thus, by strong induction, we have $a_n < 2^n$ for all $n \in \mathbb{N}$. □

Theorem (Theorem 5)

Let a_n be a sequence defined recursively as: $a_1 = 1$, $a_2 = 3$, $a_n = a_{n-2} + 2a_{n-1}$, for all $n \geq 3$. The value of a_n is odd for all $n \in \mathbb{N}$.

Proof (Proof of Theorem 5)

Suppose $P(n) : a_n$ is odd, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

Basis step:

Proof (Proof of Theorem 5)

Suppose $P(n) : a_n$ is odd, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

Basis step: Observe that $a_1 = 1$, $a_2 = 3$, and $a_3 = 7$, hence a_1, a_2, and a_3 are odd. Thus, $P(1)$, $P(2)$, $P(3)$ are true.

Inductive step:

Proof (Proof of Theorem 5)

Suppose $P(n) : a_n$ is odd, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Inductive step: Suppose $k \in \mathbb{N}$ and assume that $P(1), P(2), \ldots, P(k)$ are true, i.e., a_1, a_2, ..., a_k are odd. We must show that $P(k+1) : a_{k+1}$ is odd is also true. Observe that:

$$\begin{aligned} a_{k+1} &= a_{k-1} + 2a_k \text{ (sequence definition of } a_n\text{)} \\ &= \end{aligned}$$

Proof (Proof of Theorem 5)

Suppose $P(n) : a_n$ is odd, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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$$\begin{aligned} a_{k+1} &= a_{k-1} + 2a_k \text{ (sequence definition of } a_n\text{)} \\ &= (2p+1) + 2(2q+1), \text{ for some } p, q \in \mathbb{Z} \\ &\quad \text{since } a_{k-1} \text{ and } a_k \text{ are odd according to the inductive hypothesis} \\ &= \end{aligned}$$

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Suppose $P(n) : a_n$ is odd, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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$$\begin{aligned} a_{k+1} &= a_{k-1} + 2a_k \text{ (sequence definition of } a_n\text{)} \\ &= (2p+1) + 2(2q+1)\text{, for some } p, q \in \mathbb{Z} \\ &\quad \text{since } a_{k-1} \text{ and } a_k \text{ are odd according to the inductive hypothesis} \\ &= 2p + 1 + 4q + 2 = 2p + 4q + 2 + 1 \\ &= \end{aligned}$$

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Suppose $P(n) : a_n$ is odd, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Inductive step: Suppose $k \in \mathbb{N}$ and assume that $P(1), P(2), \ldots, P(k)$ are true, i.e., a_1, a_2, ..., a_k are odd. We must show that $P(k+1) : a_{k+1}$ is odd is also true. Observe that:

$$\begin{aligned} a_{k+1} &= a_{k-1} + 2a_k \text{ (sequence definition of } a_n\text{)} \\ &= (2p+1) + 2(2q+1), \text{ for some } p, q \in \mathbb{Z} \\ &\quad \text{since } a_{k-1} \text{ and } a_k \text{ are odd according to the inductive hypothesis} \\ &= 2p + 1 + 4q + 2 = 2p + 4q + 2 + 1 \\ &= 2(p + 2q + 1) + 1. \end{aligned}$$

This means a_{k+1} is odd, therefore $P(k+1) : a_{k+1}$ is odd is true.

Proof (Proof of Theorem 5)

Suppose $P(n) : a_n$ is odd, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

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Inductive step: Suppose $k \in \mathbb{N}$ and assume that $P(1), P(2), \ldots, P(k)$ are true, i.e., a_1, a_2, ..., a_k are odd. We must show that $P(k+1) : a_{k+1}$ is odd is also true. Observe that:

$$\begin{aligned} a_{k+1} &= a_{k-1} + 2a_k \text{ (sequence definition of } a_n\text{)} \\ &= (2p+1) + 2(2q+1), \text{ for some } p, q \in \mathbb{Z} \\ &\quad \text{since } a_{k-1} \text{ and } a_k \text{ are odd according to the inductive hypothesis} \\ &= 2p + 1 + 4q + 2 = 2p + 4q + 2 + 1 \\ &= 2(p + 2q + 1) + 1. \end{aligned}$$

This means a_{k+1} is odd, therefore $P(k+1) : a_{k+1}$ is odd is true.

We have completed the basis step and the inductive step. Thus, by strong induction, we have a_n is odd for all $n \in \mathbb{N}$. □

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Every integer $n > 1$ can be expressed as:

1. the product of two or more prime numbers, or
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- If $n = 23$, we have $23 = 23 \cdot 1$, observe that 23 is a prime number.
- If $n = 41$, we have $41 = 41 \cdot 1$, observe that 41 is a prime number.

Proof (Proof of Theorem 6)

Suppose $P(n)$: n can be expressed as the product of two or more primes or as the product of a prime and 1. We will show that $P(n)$ is true for all integers $n \geq 2$.

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- Case 2: If $k+1$ is not a prime number, then there are integers a and b such that $k+1=a\cdot b$ and $2\leq a\leq b<k+1$. From the inductive hypothesis, we know that $P(a)$ and $P(b)$ are true, in other words:
  - a can be expressed as the product of two or more primes or as the product of a prime and 1.
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  Therefore, since $k+1=a\cdot b$, the number $k+1$ can be expressed as the product of two or more primes.

From these two cases, we conclude that $P(k+1): k+1$ can be expressed as the product of two or more primes or as the product of a prime and 1, is true. By strong induction, we have shown that $P(n): n$ can be expressed as the product of two or more primes or as the product of a prime and 1, is true for any integer $n\geq 2$. □

Contents

Exercise 2: Strong Induction

Exercise

Verify the truth of the following statements:

1. Suppose a_n is a sequence defined recursively as: $a_0 = 0$, $a_1 = 4$, and $a_n = 6a_{n-1} - 5a_{n-2}$, for $n \geq 2$. The value of a_n satisfies the equation $a_n = 5^n - 1$ for all integers $n \geq 1$.
2. Suppose b_n is a sequence defined recursively as: $b_1 = 1$, $b_2 = 2$, and $b_n = \frac{1}{2}(b_{n-1} + b_{n-2})$, for $n \geq 3$. The value of b_n satisfies $1 \leq b_n \leq 2$ for all $n \in \mathbb{N}$.
3. Any whole number of cents of at least 12 cents can be obtained using 4 ¢ and 5 ¢ coins.

Solve following problems:

4. A kingdom uses galleon as its currency. Determine which amounts of galleon can be obtained from the combination of 2 galleons and 5 galleons (other that 2 galleons and 5 galleons themselves).