Mathematical Induction
Part 1: Ordinary Induction

Mathematical Logic – First Term 2023-2024

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SoC Tel-U

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Acknowledgements

This slide is compiled using the materials in the following sources:

1. *Discrete Mathematics and Its Applications* (Chapter 5), 8th Edition, 2019, by K. H. Rosen (primary reference).
2. *Discrete Mathematics with Applications* (Chapter 5), 5th Edition, 2018, by S. S. Epp.
3. Discrete Mathematics 1 (2012) slides at Fasilkom UI by B. H. Widjaja.
4. Discrete Mathematics 1 (2010) slides at Fasilkom UI by A. A. Krisnadhi.

Some figures are excerpted from those sources. This slide is intended for internal academic purpose in SoC Telkom University. No slides are ever free from error nor incapable of being improved. Please convey your comments and corrections (if any) to `<pleasedontspam>@telkomuniversity.ac.id`.

Contents

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Motivation

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Motivation

- From the previous discussion, we've seen several proof methods, i.e.: direct proof, indirect proof by contraposition, as well as indirect proof by contradiction.
- In this slide, we're going to discuss a method of proof associated with the set of natural numbers, i.e., $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$.
- For brevity, we denote $\{0, 1, 2, \ldots\}$ by $\mathbb{N}_0$ and $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$.

Some theorems associated with $\mathbb{N}_0$ or $\mathbb{N}$ can easily be proven using direct proof or indirect proof (by contraposition or by contradiction), e.g.:

Theorem

For any non-negative integer n, n is even if and only if $5n + 3$ is odd.

Theorem

If n is a non-negative integer, then $n^2 + 1 \geq 2n$.

Some theorems associated with $\mathbb{N}_0$ or $\mathbb{N}$ can easily be proven using direct proof or indirect proof (by contraposition or by contradiction), e.g.:

Theorem

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Theorem

If n is a non-negative integer, then $n^2 + 1 \geq 2n$.

However, other theorems are "more difficult" (or somehow it seems impossible) to be proven using direct proof, e.g.:

Theorem

If n is a non-negative integer, then $2^{n-1} \leq n!$.

"Verifying" the Truth by Examples

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If n is a non-negative integer, then $2^{n-1} \leq n!$.

The above theorem can be "verified" by checking its truth for any integer $n \geq 0$.

n	2^{n-1}	$n!$	$2^{n-1} \leq n!$
0			

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0	$\frac{1}{2}$		

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0	$\frac{1}{2}$	1	T
1	1	1	T
2	2	2	T
3	4	6	

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1	1	1	T
2	2	2	T
3	4	6	T
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3	4	6	T
4	8	24	

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n	2^{n-1}	$n!$	$2^{n-1} \leq n!$
0	$\frac{1}{2}$	1	T
1	1	1	T
2	2	2	T
3	4	6	T
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5			

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0	$\frac{1}{2}$	1	T
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4	8	24	T
5	16		

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4	8	24	T
5	16	120	

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2	2	2	T
3	4	6	T
4	8	24	T
5	16	120	T
$\vdots$	$\vdots$	$\vdots$	$\vdots$

However, examples cannot be used as a solid justification for the truth of the theorem, because they only provide some truth values of some elements, **not all truth values for all elements**.

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- $\forall n P(n)$
- $\forall n (n \geq a \rightarrow P(n))$, for some $a \in \mathbb{N}_0$

n is a variable in $\mathbb{N}_0$ or $\mathbb{N}$.

Remark

Mathematical induction can only be used for proving mathematical statements pertaining to $\mathbb{N}_0$ or $\mathbb{N}$, or mathematical statements over some domains which has "similar" structure with one of those domains ($\mathbb{N}_0$ or $\mathbb{N}$).

The Structure and Analogy of (Ordinary) Mathematical Induction

- How do we prove mathematical statements by mathematical induction?
- Suppose we have a statement of the form: $\forall n P(n)$, where n is a variable over $\mathbb{N}_0$.

(Ordinary) Mathematical Induction

In order to prove that $\forall n P(n)$ is true, then we have to:

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1. Prove that $P(0)$ is true, because 0 is the smallest element in $\mathbb{N}_0$. This step is called the basis step.

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2. Prove that for any integer $k \geq 0$, if $P(k)$ is true, then $P(k+1)$ is also true. This step is called the inductive step.

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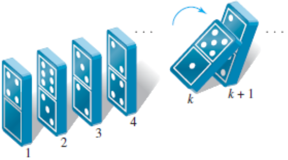
Mathematical induction can be visualized as "domino effect".



Why does mathematical induction work?

- Observe that the two steps in mathematical induction is sufficient for proving that $\forall n P(n)$ is true.
- If the inductive step can be shown to be true, then we have the conditional statement $P(k) \rightarrow P(k+1)$ which is true for any k. This is equivalent with the predicate formula $\forall k (P(k) \rightarrow P(k+1))$, where the domain of k is $\mathbb{N}_0$.

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- Observe that the two steps in mathematical induction is sufficient for proving that $\forall n P(n)$ is true.
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- Therefore, if $P(0)$ is true, using the fact that $P(0) \to P(1)$ is true and modus ponens, we have $P(1)$ is true.
- Then, since $P(1)$ is true, using the fact that $P(1) \to P(2)$ is true and modus ponens, we have

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- Then, since $P(1)$ is true, using the fact that $P(1) \rightarrow P(2)$ is true and modus ponens, we have $P(2)$ is true.
- Continuing along these lines, we see that $P(n)$ is true for any n.
- In the conditional statement $P(k) \rightarrow P(k+1)$, $P(k)$ is called the <u>induction hypothesis</u>.
- The basis step in an induction <u>may not be started</u> from 0.

Contents

Proofs Using (Ordinary) Mathematical Induction

Theorem (Theorem 1)

If n is a positive integer, then $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Proof (Proof of Theorem 1)

Suppose $P(n)$ is the statement $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, where $n \in \mathbb{N}$. We will show that $P(n)$ is true for all $n \in \mathbb{N}$.

Basis step:

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$P(k)\equiv 1+2+3+\cdots+k=\frac{k(k+1)}{2}$ is true. We must show that

$P(k+1)\equiv$

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$P(k+1) \equiv 1+2+3+\cdots+k+k+1 = \frac{(k+1)(k+2)}{2}$ is also true.
Observe that by adding $k+1$ to each side of $P(k)$, we have:

$$(1+2+3+\cdots+k)+k+1 \quad = \quad \frac{k(k+1)}{2}$$

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$$
\begin{aligned}
(1+2+3+\cdots+k)+k+1 &= \frac{k(k+1)}{2}+(k+1) \text{ (by inductive hypothesis)} \\
&=
\end{aligned}
$$

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$$\begin{aligned}(1+2+3+\cdots+k)+k+1 &= \frac{k(k+1)}{2}+(k+1) \text{ (by inductive hypothesis)}\\ &= \frac{k(k+1)}{2}+\frac{2k+2}{2}=\frac{k^2+3k+2}{2}\\ &= \frac{(k+1)(k+2)}{2}.\end{aligned}$$

Therefore $P(k+1)$ is also true.
We have completed the basis step and the inductive step. Thus, by mathematical induction, we have $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ for any $n\in\mathbb{N}$. □

Theorem (Theorem 2)

For any integer $n \geq 1$, then $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Proof (Proof of Theorem 2)

Suppose $P(n)$ is the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, where $n \in \mathbb{N}$.
We will show that $P(n)$ is true for all $n \in \mathbb{N}$.
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Basis step: Observe that $P(1)$ is the statement $1 = 1^2$. Obviously $P(1)$ is true.
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Suppose $P(n)$ is the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, where $n \in \mathbb{N}$.
We will show that $P(n)$ is true for all $n \in \mathbb{N}$.
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Therefore $P(k+1)$ is also true.
We have completed the basis step and the inductive step. Thus, by mathematical induction, we have $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for any $n \in \mathbb{N}$. □

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For any integer $n \geq 3$, then $2n + 1 < 2^n$.

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Suppose $P(n) \equiv 2n + 1 < 2^n$, where $n \in \mathbb{N}$ and $n \geq 3$. We will show that $P(n)$ is true for all integers $n \geq 3$.

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Contents

Exercise 1: (Ordinary) Mathematical Induction

Exercise

1. For what values of integers n does the inequality $n^2 < 2^n$ apply? Explain your answer. (Hint: you may use the result in Theorem 3, i.e., $2n + 1 < 2^n$ for all integers $n \geq 3$).
2. For what values of integers does the inequality $2^n < n!$ apply? Explain your answer.
3. Let $x > 0$ be a real number. For what values of n does the inequality $(1 + x)^n \geq 1 + nx$ apply? Justify your answer.
4. For what values of integers n is the expression $n^3 - n$ divisible by 3? Explain your answer.